Fractional Super Lie Algebras and Groups

H. Ahmedov[1], A. Yildiz[1] and Y. Ucan[2]

1. Feza Gürsey Institute, P.O. Box 6, 81220, Çengelköy, Istanbul, Turkey [1].
2. Yildiz Technical University, Department of Mathematics, Besiktas, Istanbul, Turkey.

Abstract

n$\underline{\text{th}}$ root of a Lie algebra and its dual (that is fractional supergroup) based on the permutation group S_n invariant forms is formulated in the Hopf algebra formalism. Detailed discussion of S_3-graded sl(2) algebras is done.

1. Introduction

To arrive at a superalgebra one adds new elements Q_α to generators X_j of the corresponding Lie algebra and define relations

$$\{Q_\alpha, Q_\beta\} = b^j_{\alpha\beta} X_j. \tag{1}$$

Observing that the anticommutator in the above relation is invariant under the cyclic Z_2 or permutation S_2 groups we can look for possible generalization of the supersymmetry by using S_n or Z_n invariant structures instead of the anticommutator. For example if $n = 3$ instead of (1) one has cubic relations

$$Q_\alpha Q_\beta Q_\gamma + Q_\gamma Q_\alpha Q_\beta + Q_\beta Q_\gamma Q_\alpha = b^j_{\alpha\beta\gamma} X_j \tag{2}$$

which is Z_3 invariant and

$$Q_\alpha \{Q_\beta, Q_\gamma\} + Q_\beta \{Q_\alpha, Q_\gamma\} + Q_\gamma \{Q_\alpha, Q_\beta\} = b^j_{\alpha\beta\gamma} X_j \tag{3}$$

which is S_3 invariant. From the above relations only (3) appears to be consistent at the co-algebra level. Sometimes we will use the term fractional superalgebras for S_n-graded algebras with $n = 3, 4, \ldots ...$ with fractional super groups being their dual.

Fractional super algebras based on S_n invariant forms were first introduced in [1, 2]. In the present paper we put this construction in the Hopf algebra context and define their dual, that is fractional supergroups. There are many reasons for doing that. In the formulation of superalgebras one can use either geometric (See for example [3]) or algebraic [4] approaches (See also [5] for a comparison). As for fractional superalgebras geometric approach seems to be insufficient. This situation is similar to the theory of quantum algebras, where we have to work with universal enveloping algebras rather than with Lie algebras [6]. Moreover, having a fractional superalgebra in hand we can define fractional supergroups by taking the dual of the

[1] E-mail : hagi@gursey.gov.tr

former. And at last having put fractional superalgebras in the Hopf algebra context we can use well developed representation theory of the latter in the construction of representations of fractional superalgebras.

There are another approaches to fractional supersymmetry in literature [7, 8, 9, 10, 11, 12]. For example one can arrive at fractional supergroups by using quantum groups at roots of unity [13].

The plan of the paper is as follows. To make the treatment reasonably self-consistent, in section 2 we give a formulation of super algebras and groups in the Hopf algebra formalism. In section 3 we define fractional super algebras and discuss the structure of their dual (fractional super groups). Section 4 is devoted to the detailed discussion of S_3-graded sl(2) algebras.

2. Preliminaries on super algebras

Let U(g) be the universal enveloping algebra of a Lie algebra g generated by X_j, $j = 1, \ldots \mathrm{dim(g)}$ with

$$[X_i, X_j] = \sum_{k=1}^{\mathrm{dim(g)}} c_{ij}^k X_k, \tag{4}$$

where c_{ij}^k are the structure constants of the Lie algebra g. The Hopf algebra structure of U(g) is given by the co-multiplication $\Delta : \mathrm{U(g)} \to \mathrm{U(g)} \otimes \mathrm{U(g)}$, co-unit $\varepsilon : \mathrm{U(g)} \to \mathrm{C}$ and antipode $S : \mathrm{U(g)} \to \mathrm{U(g)}$:

$$\Delta(X_j) = X_j \otimes 1 + 1 \otimes X_j, \quad \varepsilon(X_j) = 0, \quad S(X_j) = -X_j \tag{5}$$

We can extend the Hopf algebra U(g) by adding elements Q_α, $\alpha = 1, \ldots, N$ and K with relations

$$\{Q_\alpha, Q_\beta\} = \sum_{j=1}^{\mathrm{dim(g)}} b_{\alpha\beta}^j X_j \tag{6}$$

$$[Q_\alpha, X_j] = \sum_{\beta=1}^{\mathrm{N}} a_{\alpha\beta}^j Q_\beta, \tag{7}$$

$$KQ_\alpha = -Q_\alpha K, \qquad K^2 = 1 \tag{8}$$

where $b_{\alpha\beta}^j$ and $a_{\alpha\beta}^j$ are the structure coefficients satisfying the super Jacobi identities. This algebra which we denote by $\mathrm{U_2^N(g)}$ can also be equipped with a Hopf algebra structure by defining

$$\Delta(Q_\alpha) = Q_\alpha \otimes 1 + K \otimes Q_\alpha, \quad \Delta(K) = K \otimes K, \tag{9}$$

$$\varepsilon(Q_j) = 0, \quad \varepsilon(K) = 1, \quad S(Q_j) = Q_j K, \quad S(K) = K. \tag{10}$$

The dual of $\mathrm{U_2^N(g)}$ is the Hopf algebra $\mathrm{A_2^N(G)} = \mathrm{C^\infty(G)} \times \Lambda_2^{\mathrm{N}}$, where $\mathrm{C^\infty(G)}$ is the algebra of infinite differentiable functions on a Lie group G and Λ_2^N is the algebra over the field of complex numbers generated by θ_α, $j = 1, \ldots, N$ and λ with relations

$$\{\theta_\alpha, \theta_\beta\} = 0, \quad \{\lambda, \theta_\alpha\} = 0, \quad \lambda^2 = 1. \tag{11}$$

The operations Δ, ε and S in $\mathrm{A^N(G)}$ depend on the value of structure constants c^k_{ij}, $b^j_{\alpha\beta}$ and $a^j_{\alpha\beta}$.

For example if $\mathrm{G} = \mathrm{C^N}$ then the formulas

$$\Delta(\theta_\alpha) = \theta_\alpha \otimes 1 + \lambda \otimes \theta_\alpha, \;\; \Delta(\lambda) = \lambda \otimes \lambda, \;\; \Delta(z_\alpha) = z_\alpha \otimes 1 + 1 \otimes z_\alpha + \lambda\theta_\alpha \otimes \theta_\alpha, \quad (12)$$

$$\varepsilon(\theta_\alpha) = 0, \quad \varepsilon(\lambda) = 1, \quad \varepsilon(z_\alpha) = 0 \quad (13)$$

$$S(\theta_\alpha) = -\lambda\theta_\alpha, \quad S(\lambda) = \lambda, \quad S(z_\alpha) = -z_\alpha \quad (14)$$

define the super N-dimensional translation group. The corresponding super algebra is defined by

$$\{Q_\beta, Q_\alpha\} = \delta_{\beta\alpha} P_\alpha, \quad [X_\beta, X_\alpha] = 0, \quad [Q_\beta, X_\alpha] = 0. \quad (15)$$

3. Fractional superalgebras and supergroups

To arrive at cubic root of a lie algebra g we have to replace S_2 invariant form in (6) by S_3 invariant one. Consequently we define an algebra generated by X_j, $j = 1, \ldots, dim(g)$ and Q_α, K, $\alpha = 1, \ldots, N$ satisfying the relations (4) and

$$\{Q_\alpha, Q_\beta, Q_\gamma\} = b^j_{\alpha\beta\gamma} X_j, \quad (16)$$

$$[Q_\alpha, X_j] = a^j_{\alpha\beta} Q_\beta, \quad (17)$$

and

$$K Q_\alpha = q Q_\alpha K, \quad q^3 = 1, \quad K^3 = 1, \quad (18)$$

where

$$\{Q_\alpha, Q_\beta, Q_\gamma\} \equiv Q_\alpha\{Q_\beta, Q_\gamma\} + Q_\beta\{Q_\alpha, Q_\gamma\} + Q_\gamma\{Q_\alpha, Q_\beta\} \quad (19)$$

is S_3 invariant form. We denote this algebra by the symbol $U^N_3(g)$ with the lower index indicating the degree of grading. One can check that the above algebra is compatible with the co-algebra structure and antipode given by the formulas

$$\Delta(Q_\alpha) = Q_\alpha \otimes 1 + K \otimes Q_\alpha, \quad \Delta(K) = K \otimes K, \quad (20)$$

$$\varepsilon(Q_j) = 0, \quad \varepsilon(K) = 1, \quad S(Q_j) = -K^2 Q_j, \quad S(K) = K^2. \quad (21)$$

For example let us verify the consistency of the comultiplication Δ with (16). Since Δ is a homomorphism we have

$$\begin{aligned}\Delta(Q_\alpha Q_\beta Q_\gamma) = {} & Q_\alpha Q_\beta Q_\gamma \otimes 1 + 1 \otimes Q_\alpha Q_\beta Q_\gamma + Q_\alpha Q_\beta K \otimes Q_\gamma \\ & + Q_\alpha K Q_\gamma \otimes Q_\beta + K Q_\beta Q_\gamma \otimes Q_\alpha + Q_\alpha K^2 \otimes Q_\beta Q_\gamma \\ & + K Q_\beta K \otimes Q_\alpha Q_\gamma + K^2 Q_\gamma \otimes Q_\alpha Q_\beta \end{aligned} \quad (22)$$

Using (18) we get

$$\sum_{(\alpha\beta\gamma)\in S_3} (Q_\alpha Q_\beta K \otimes Q_\gamma + Q_\alpha K Q_\gamma \otimes Q_\beta + K Q_\beta Q_\gamma \otimes Q_\alpha) = 0 \quad (23)$$

and

$$\sum_{(\alpha\beta\gamma)\in S_3} (Q_\alpha K^2 \otimes Q_\beta Q_\gamma + K Q_\beta K \otimes Q_\alpha Q_\gamma + K^2 Q_\gamma \otimes Q_\alpha Q_\beta) = 0. \tag{24}$$

Thus we have shown that

$$\sum_{(\alpha\beta\gamma)\in S_3} \Delta(Q_\alpha Q_\beta Q_\gamma) = \sum_{(\alpha\beta\gamma)\in S_3} (Q_\alpha Q_\beta Q_\gamma \otimes 1 + 1 \otimes Q_\alpha Q_\beta Q_\gamma) \tag{25}$$

which together with the comultiplication rule (5) for the generators X_j implies the consistency of the comultiplication (20) with the relation (16).

To define structure constants $b^j_{\alpha\beta\gamma}$ and $a^j_{\alpha\beta}$ we have to derive identities involving commutator and S_3 invariant form. One can check that relations

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0, \tag{26}$$

$$[A,\{B,C,D\}] + \{[B,A],C,D\} + \{B,[C,A],D\} + \{B,C,[D,A]\} = 0 \tag{27}$$

and

$$[A,\{B,C,D\}] + [B,\{A,C,D\}] + [C,\{B,A,D\}] + [D,\{B,C,A\}] = 0 \tag{28}$$

are satisfied identically. For example we verify the identity (27). Let $B = A_1$, $C = A_2$ and $D = A_3$. Then

$$[A,\{A_1,A_2,A_3\}] = \sum_{ijk\in S_3} ([A,A_i]A_jA_k + A_iA_j[A,A_k] + A_i[A,A_j]A_k). \tag{29}$$

Combining terms (123), (132) from the first sum on the right hand side of the above equality, (231), (321) from the second sum and (213), (312) from the third sum we get $\{[A,A_1],A_2,A_3\}$. In a similar fashion we obtain $\{[A,A_2],A_1,A_3\}$ and $\{[A,A_3],A_2,A_1\}$. Thus

$$[A,\{A_1,A_2,A_3\}] = \{[A,A_1],A_2,A_3\} + \{[A,A_2],A_1,A_3\} + \{[A,A_3],A_2,A_1\} \tag{30}$$

which is the identity (27).

The one given by (26) is the usual Jacobi identity. Inserting

$$A = X_i, \quad B = X_j, \quad C = Q_\alpha \tag{31}$$

into (26) and using (17) and (4) we get

$$\sum_{\sigma=1}^{N} (a^i_{\alpha\sigma} a^j_{\sigma\beta} - a^j_{\alpha\sigma} a^i_{\sigma\beta}) = \sum_{k=1}^{dim(g)} c^k_{ij} a^k_{\alpha\beta}. \tag{32}$$

Comparing the above relation with (4) we conclude that the $N \times N$ matrices $a^j \equiv (a^j_{\alpha\beta})^N_{\alpha,\beta=1}$ define an N-dimensional representation of a given Lie algebra. There are different possibilities in the choice of this representation. For example if $g = sl(2)$ and $N = 2$ we can either use the scalar representation $a^j_{\alpha\beta} = 0$ or the spinor one in which $a^j_{\alpha\beta}$ are the Pauli matrices. Consequently for fixed Lie algebra g and N we can define different super fractional algebras. To be more precise one has to add additional index in the notation $U^N_3(g)$

which reflects the transformation law of super generators Q_α with respect to a given Lie algebra g. However for the sake of simplicity we will not do it. Detailed discussion of this non uniqueness is done in the next section where we consider fractional super algebras $sl(2)$.

Let us now consider restrictions on structure coefficients coming from the other identities. Inserting

$$A = X_k, \quad B = Q_\alpha, \quad C = Q_\beta, \quad D = Q_\gamma \tag{33}$$

into the identity (27) and

$$A = Q_\sigma, \quad B = Q_\alpha, \quad C = Q_\beta, \quad D = Q_\gamma \tag{34}$$

into (28) and using (16), (17) we arrive at relations

$$\sum_{\sigma=1}^{N} (a^k_{\alpha\sigma} b^i_{\sigma\beta\gamma} + a^k_{\beta\sigma} b^i_{\sigma\alpha\gamma} + a^k_{\gamma\sigma} b^i_{\sigma\beta\alpha}) = \sum_{j=1}^{dimg} c^i_{jk} b^j_{\alpha\beta\gamma} \tag{35}$$

and

$$\sum_{k=1}^{dimg} (b^k_{\alpha\beta\gamma} a^k_{\sigma\tau} + b^k_{\sigma\alpha\beta} a^k_{\gamma\tau} + b^k_{\gamma\sigma\alpha} a^k_{\beta\tau} + b^k_{\beta\gamma\sigma} a^k_{\alpha\tau}) = 0. \tag{36}$$

Now we define fractional super groups. Let $x = \{x_{nm}\}$ be the matrix representing a Lie group G and $A(G)$ be the algebra of polynomials on G. It is known that $A(G)$ is the Hopf algebra which is in non degenerate duality with the universal enveloping algebra $U(g)$ [14]. In general the number of group elements x_{nm} is more than the number of generators X_j in the corresponding Lie algebra g. This is due to the fact that there may be some restrictions on the matrix representing a Lie group. For example if $G = SL(2)$ we have two by two matrix with determinant equal to 1. The number of independent group parameters is equal to the number of generators of $sl(2)$. Explicitly one can define these parameters by using some decomposition (Gauss, Cartan, Iwasawa and so on). In a similar way for an arbitrary matrix Lie group we can resolve restrictions imposed on the elements x_{nm} and obtain independent group parameters x_j with duality relations

$$\langle x_i, X_j \rangle = \delta_{ij}, \tag{37}$$

where X_j are the generators of the corresponding Lie algebra. However in general $A(G)$ in terms of these new parameters will not be the polynomial algebra. It appears that in the Hopf algebra formalism it is more convenient to work with elements x_{nm}. Instead of solving restrictions imposed on these elements one defines new generators X_{nm} with some restrictions. For example if $g = sl(2)$ we define four generators with the restriction $X_{11} + X_{22} = 0$.

To construct the dual algebra to a fractional super algebra $U_3^N(g)$ we have to introduce new parameters θ_α, $\alpha = 1, \ldots, N$ and λ corresponding to the fractional super generators Q_α and K. The duality relations are given by the following formulas

$$\langle \theta_\alpha, Q_\beta \rangle = \delta_{\alpha\beta}, \qquad \langle \lambda, K \rangle = q, \quad \langle x_{nm}, K \rangle = \delta_{nm} \tag{38}$$

with all other linear relations being zero. Recall the property of the duality relations [14]

$$\langle ab, \phi\rangle = \sum_j \langle a, \phi_j\rangle\langle b, \phi'_j\rangle \tag{39}$$

with

$$\Delta(\phi) = \sum_j \phi_j \otimes \phi'_j. \tag{40}$$

Here ϕ and a, b are elements of a Hopf algebra and its dual. Inserting in (39) $a = \theta_\alpha$, $b = \lambda$ and $\phi = Q_\alpha$ and using (20), (38) we get

$$\lambda\theta_\alpha = q\theta_\alpha\lambda \tag{41}$$

Taking $a = x_{nm}$, $b = \lambda$ and $\phi = X_{nm}$ we conclude that elements x_{nm} commute with λ. The choice $a = \lambda^2$, $b = \lambda$ and $\phi = K$ implies $\langle\lambda^3, K\rangle = 1$. Since λ^3 cannot be proportional to the diagonal elements x_{nn} ($\langle\lambda^3, X_{nn}\rangle = 0$) we have

$$\lambda^3 = 1. \tag{42}$$

The above condition can be shown to imply the comultiplication

$$\Delta(\lambda) = \lambda \otimes \lambda. \tag{43}$$

To make (41) and (43) compatible we have to define

$$\Delta(\theta_\alpha) = \sum_{\beta=1}^{N} \theta_\beta \otimes d_{\beta\alpha} + \lambda \otimes \theta_\beta + \cdots, \tag{44}$$

where $d = \{d_{\alpha\beta}\}$ is a N dimensional representation of a Lie group G under consideration and $\cdots$ denotes the combination of terms consisting of 4, 7, 10 and so on generators θ_α. Using (44), (38) and (39) after some algebra we get

$$\{\theta_\alpha, \theta_\beta, \theta_\gamma\} = 0 \tag{45}$$

In a similar way the commutativity of x_{nm} with λ and (43) imply

$$\Delta(x_{nm}) = \sum_k x_{nk} \otimes x_{km} + \cdots, \tag{46}$$

where $\cdots$ denotes the combination of terms consisting of 3, 6, 9 and so on generators θ_α. Using (46) and (39) we conclude that elements x_{nm} commute with θ_α.

Let us denote the algebra generated by θ_α, $\alpha = 1, \ldots N$ and λ satisfying (41), (42) and (45) by Λ_3^N and the direct product algebra $A(G) \times \Lambda_3^N$ by $A_3^N(G)$. This algebra is in nondegenerate duality with a Hopf algebra $U_3^N(g)$. We call $A_3^N(G)$ fractional supergroup. Using the properties

$$\varepsilon(a) = \langle a, 1\rangle \tag{47}$$

and

$$\langle S(a), \phi\rangle = \langle a, S(\phi)\rangle \tag{48}$$

of the duality relations we get the counit operation

$$\varepsilon(x_{nm}) = \delta_{nm}, \quad \varepsilon(\theta_\alpha) = 0, \quad \varepsilon(\lambda) = 1. \tag{49}$$

and the antipode

$$S(\lambda) = \lambda^2. \tag{50}$$

Using the properties of duality relations and Hopf algebra axioms one can derive unknown terms in (44) and (46) and antipodes $S(x_{nm})$, $S(\theta_\alpha)$. These calculations depend on structure constants c^i_{jk}, $a^j_{\alpha\beta}$ and $b^j_{\alpha\beta\gamma}$. We demonstrate this construction on the explicit examples which will be given later.

Before closing this section we define S_n-graded Lie algebras and groups. This can be done in the same way as S_3 case. For this one has to use S_n invariant form

$$\{Q_{\alpha_1}, Q_{\alpha_2}, \ldots, Q_{\alpha_n}\} = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n \in S_n} Q_{\alpha_1} Q_{\alpha_2} \cdots Q_{\alpha_n}, \tag{51}$$

where summation runs over all permutations of S_n. Instead of (16) and (18) we then have

$$\{Q_{\alpha_1}, Q_{\alpha_2}, \ldots, Q_{\alpha_n}\} = b^j_{\alpha_1\alpha_2,\ldots,\alpha_n} X_j \tag{52}$$

and

$$KQ_\alpha = qQ_\alpha K, \quad q^n = 1, \quad K^n = 1 \tag{53}$$

such that

$$\sum_{\sigma=1}^{N} \sum_{(\alpha_1,\ldots,\alpha_n)\in Z_n} a^k_{\alpha_1\sigma} b^i_{\sigma_2\ldots\alpha_n} = \sum_{j=1}^{dimg} c^i_{jk} b^j_{\alpha_1\alpha_2\ldots\alpha_n} \tag{54}$$

and

$$\sum_{k=1}^{dim(g)} \sum_{(\alpha_1,\ldots,\alpha_{n+1})\in Z_{n+1}} b^k_{\alpha_1\ldots\alpha_n} a^k_{\alpha_{n+1}\tau} = 0. \tag{55}$$

The multiplication and counit in $U^N_n(g)$ are similar to that in $U^N_2(g)$ or $U^N_3(g)$ while the antipode is given by

$$S(Q_\alpha) = -K^{n-1} Q_\alpha, \quad S(K) = K^{n-1}. \tag{56}$$

The fractional super group is the algebra $A^N_n(G) = A(G) \times \Lambda^N_n$ where Λ^N_n is the algebra generated by θ_α, $\alpha = 1, \ldots, N$, λ with relations

$$\{\theta_{\alpha_1}, \theta_{\alpha_2}, \ldots, \theta_{\alpha_n}\} = 0, \qquad \alpha_k \in 1, 2, \ldots, N \tag{57}$$

and

$$\lambda\theta_\alpha = q\theta_\alpha\lambda, \quad \lambda^n = 1. \tag{58}$$

Co-algebra operations and antipode in $A^N_n(G)$ depend on the structure constants c^k_{ij}, $a^j_{\alpha\beta}$ and $b^j_{\alpha_1\alpha_2,\ldots,\alpha_n}$. As an example let us consider the fractional super algebra

$$Q^n = P, \quad [X, Q] = 0. \tag{59}$$

Since we have only one super element Q the S_n invariant form is equal up to the multiple to Q^n. The corresponding fractional group is generated by θ, z and λ such that

$$\theta^n = 0, \quad \lambda^n = 1, \quad \lambda\theta = q\theta\lambda, \quad q^n = 1 \tag{60}$$

with z being commutative with θ and λ. The duality relations are

$$\langle Q, \theta\rangle = 1, \quad \langle X, z\rangle = 1, \quad \langle K, \lambda\rangle = q. \tag{61}$$

Using properties of duality relations we arrive at the following coalgebra structure

$$\Delta(\theta) = \theta \otimes 1 + \lambda \otimes \theta \tag{62}$$

$$\Delta(z) = z \otimes 1 + 1 \otimes z + \sum_{k=1}^{n-1} \frac{\lambda^{n-k}\theta^k \otimes \theta^{n-k}}{(q:q)_k (q:q)_{n-k}}, \tag{63}$$

$$\varepsilon(\theta) = 0, \quad \varepsilon(\lambda) = 1 \tag{64}$$

and

$$S(\theta) = -\lambda^{n-1}\theta, \quad S(\lambda) = \lambda^{n-1} \tag{65}$$

where

$$(q;q)_k = \prod_{j=1}^{k} (1 - q^j) \tag{66}$$

4. S_3 graded super algebras $sl(2)$

From the commutation relations

$$[X_1, X_2] = X_3, \quad [X_3, X_1] = 2X_1, \quad [X_3, X_2] = -2X_2 \tag{67}$$

for the algebra $sl(2)$ we read

$$c_{12}^3 = 1, \quad c_{31}^1 = 2, \quad c_{32}^2 = -2. \tag{68}$$

For given N the matrix $a^j = \{a^j_{\alpha\beta}\}$ due to (32) is an arbitrary N-dimensional representation of $sl(2)$. The solution of (35) and (36) for $b^j_{\alpha\beta\gamma}$ is fully determined by this representation. Since $b^j_{\alpha\beta\gamma}$ is symmetric in α, β and γ through (16) the number of unknown coefficients for the sl(2) case is $N(N+1)(N+2)/2$. On the other hand, equation (35) which is symmetric in $\alpha,\ \beta,\ \gamma$ gives $3N(N+1)(N+2)/2$ equations and equation (36) which is symmetric in $\alpha,\ \beta,\ \gamma,\ \sigma$ gives $N^2(N+1)(N+2)(N+3)/24$ equations. Although the system seems overdetermined there are solutions some of which will be given below. We consider $N = 1,\ 2$ and 3 fractional super generalizations of $sl(2)$ at $n = 3$, that is $q = e^{i\frac{2\pi}{3}}$.

A1. $N = 1$ fractional super $sl(2)$

We have one super generator Q_1 which can transform as scalar only. Therefore $a^j_{11} = 0$. Inserting it in the relations (35) and (36) we get $b^j_{111} = 0$. These structure constants imply that the fractional super algebra $U^1_3(sl(2))$ is the direct product of the universal enveloping algebra $U(sl(2))$ and the Hopf algebra generated by Q_1 and K satisfying the relations

$$KQ_1 = qQ_1K, \quad Q_1^3 = 0, \quad K^3 = 1 \tag{69}$$

and the co-algebra operations (20) and (21). The fractional super group $A^3_1(SL(2))$ is the direct product of the Hopf algebras $A(SL(2))$ and Λ^1_3. Recall that the Hopf algebra structure of polynomial algebra $A(SL(2))$ is given by

$$\Delta(x_{nm}) = \sum_{k=1}^{2} x_{nk} \otimes x_{km} \tag{70}$$

and

$$S(x_{11}) = x_{22}, \quad S(x_{22}) = x_{11}, \quad S(x_{12}) = -x_{12}, \quad S(x_{21}) = -x_{21}, \tag{71}$$

where two by two matrix $x = \{x_{nm}\}$ representing $SL(2)$ has determinant 1. The Hopf algebra structure of the algebra Λ_3^1 is given by the following formulas

$$\Delta(\theta_1) = \theta_1 \otimes 1 + \lambda \otimes \theta_1, \quad \Delta(\lambda) = \lambda \otimes \lambda \tag{72}$$

$$S(\theta_1) = -\lambda^2 \theta_1, \quad S(\lambda) = \lambda^2. \tag{73}$$

A2. $N = 2$ **fractional super** $sl(2)$

For $N = 2$ we have two possibilities. We can either require generators Q_1, Q_2 to transform as scalars or as spinors.

(i) In the former case we have $a^j_{\alpha,\beta} = 0$. From the relations (35) and (36) we get $b^j_{\alpha\beta\gamma} = 0$. The obtained structure constants imply that the fractional super algebra $U_3^2(sl(2))$ is the direct product of the universal enveloping algebra $U(sl(2))$ and the Hopf algebra generated by Q_1, Q_2 and K satisfying the relations

$$KQ_\alpha = qQ_\alpha K, \qquad \{Q_\alpha, Q_\beta, Q_\gamma\} = 0, \qquad K^3 = 1 \tag{74}$$

and the co-algebra operations (20) and (21). The fractional super group $A_2^3(SL(2))$ is the direct product of the Hopf algebras $A(SL(2))$ and Λ_3^2. The Hopf algebra structure of Λ_3^2 is given by the following formulas

$$\Delta(\theta_\alpha) = \theta_\alpha \otimes 1 + \lambda \otimes \theta_\alpha, \quad \Delta(\lambda) = \lambda \otimes \lambda \tag{75}$$

$$S(\theta_\alpha) = -\lambda^2 \theta_\alpha, \quad S(\lambda) = \lambda^2. \tag{76}$$

(ii) Now let us assume that Q_1 and Q_2 transform as spinors under the action of $sl(2)$. We have

$$a^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad a^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad a^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{77}$$

Equation (36) gives 10 equations for 12 unknowns

$$b^1_{111} = b^1_{112} = b^2_{122} = b^2_{222} = b^3_{111} = b^3_{222} = 0, \tag{78}$$

$$b^1_{122} = -\frac{1}{3} b^2_{111} = b^3_{112}, \quad b^1_{222} = -3b^2_{112} = 3b^3_{122} \tag{79}$$

Substituting these into (35) one finds that the only solution is $b^j_{\alpha\beta\gamma} = 0$. Thus we obtained the following fractional super algebra

$$\{Q_\alpha, Q_\beta, Q_\gamma\} = 0, \tag{80}$$

$$[Q_1, X_1] = Q_2, \quad [Q_2, X_2] = Q_1, \quad [Q_1, X_3] = Q_1, \quad [Q_2, X_3] = -Q_2. \tag{81}$$

Using the general construction given in the previous section one can define the fractional super group $A_3^1(SL(2))$ corresponding to the above fractional super algebra. $A_3^1(SL(2)$ is the algebra generated by elements x_{nm}, θ_n, $n, m = 1, 2$

and λ satisfying (57), (58) and $\det(x_{nm}) = 1$. The co-algebra operations and antipode can be shown to be given by (70), (71) and

$$\begin{aligned} \Delta(\theta_1) &= \theta_2 \otimes x_{21} + \theta_1 \otimes x_{11} + \lambda \otimes \theta_1 && (82)\\ \Delta(\theta_2) &= \theta_2 \otimes x_{22} + \theta_1 \otimes x_{12} + \lambda \otimes \theta_2 && (83) \end{aligned}$$

and

$$S(\lambda) = \lambda^2, \quad S(\theta_1) = \lambda^2(x_{21}\theta_2 - x_{22}\theta_1), \quad S(\theta_2) = \lambda^2(x_{12}\theta_1 - x_{11}\theta_2). \quad (84)$$

The duality relations are given by the formulas

$$\langle X_3, x_{nn}\rangle = (-)^{n+1}, \quad \langle X_1, x_{12}\rangle = 1, \quad \langle K, x_{nm}\rangle = \delta_{nm} \quad (85)$$
$$\langle X_2, x_{21}\rangle = 1, \quad \langle Q_\alpha, \theta_\beta\rangle = \delta_{\alpha\beta}, \quad \langle K, \lambda\rangle = q. \quad (86)$$

A3. $N = 3$ **fractional super** $sl(2)$**.**

We have three different superalgebras depending on the choice of a^j.

(i) We take $a^j_{\alpha\beta} = 0$. The relations (35) and (36) imply $b^j_{\alpha\beta\gamma=0}$. This case is similar with (i) of $A2$.

(ii) We take the vector representation

$$a^1 = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \; a^2 = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \; a^3 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad (87)$$

The substitution of

$$a^1_{21} = a^1_{32} = a^2_{12} = a^2_{23} = \sqrt{2}, \quad a^3_{11} = -2, \quad a^3_{33} = 2 \quad (88)$$

into (36) gives

$$\begin{aligned} b^1_{111} &= 3\sqrt{2}b^3_{112} = -3b^2_{113}, \\ b^1_{112} &= \sqrt{2}b^3_{122} = -2b^2_{123}, \\ b^1_{122} &= \frac{\sqrt{2}}{3}b^3_{222} = -b^2_{223}, \\ b^1_{113} &= 2\sqrt{2}b^3_{123} = -b^2_{133}, \\ b^1_{133} &= \sqrt{2}b^3_{233} = -\frac{1}{3}b^2_{333}, \\ b^1_{123} &= \frac{\sqrt{2}}{2}b^3_{223} = -\frac{1}{2}b^2_{233} \end{aligned} \quad (89)$$

and the remaining 12 parameters $b^j_{\alpha\beta\gamma}$ are zero. The substitution of (89) into (35) gives

$$b^1_{113} = -2b^1_{122} = -b^2_{133} = 2b^2_{223} = 2\sqrt{2}b^3_{123} = -\frac{2\sqrt{2}}{3}b^3_{222} \quad (90)$$

and all other $b^j_{\alpha\beta\gamma}$ are zero. Hence we have a unique extension for the vector representation of sl(2). Equations satisfied by $b^j_{\alpha\beta\gamma}$, namely (35) and (36), are

invariant under rescaling $b^j_{\alpha\beta\gamma} \to k b^j_{\alpha\beta\gamma}$ where k is any nonzero constant. The choice of this nonzero constant results only in a rescaling of the generators Q_α. Just for the sake of simplicity we choose $Q_2^3 = X_3$, i.e., $b^3_{222} = 6$. Then the fractional supersymmetric extension of sl(2) reads

$$[Q_1, X_2] = \sqrt{2} Q_2,\ [Q_1, X_3] = -2Q_1,\ [Q_2, X_1] = \sqrt{2} Q_1, \tag{91}$$

$$[Q_2, X_2] = \sqrt{2} Q_3,\ [Q_3, X_1] = \sqrt{2} Q_2,\ [Q_3, X_3] = 2Q_3 \tag{92}$$

and

$$\{Q_1, Q_1, Q_3\} = -4\sqrt{2} X_1,\ \{Q_1, Q_2, Q_2\} = 2\sqrt{2} X_1,\ \{Q_1, Q_2, Q_3\} = -2X_3, \tag{93}$$

$$\{Q_1, Q_3, Q_3\} = -4\sqrt{2} X_2,\ \{Q_2, Q_2, Q_2\} = 6X_3,\ \{Q_2, Q_2, Q_3\} = -2\sqrt{2} X_2. \tag{94}$$

Notice also that all $b^j_{\alpha\beta\gamma} = 0$ is always a solution of (35) and (36).

(iii) Assume that two of fractional super generators Q_1, Q_2 and Q_3 transform as spinors and the remaining one transforms as scalar, that is

$$a^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ a^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ a^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{95}$$

The conditions (35) and (36) imply

$$b^1_{223} = -b^2_{111} = 2b^3_{123} \tag{96}$$

with all other structure coefficients $b^j_{\alpha\beta\gamma}$ being zero. Choosing $b^1_{223} = 1$ we get the fractional super algebra given by (81) and

$$\{Q_1, Q_1, Q_3\} = -X_2, \quad \{Q_2, Q_2, Q_3\} = X_1, \quad \{Q_1, Q_2, Q_3\} = \frac{1}{2} X_3 \tag{97}$$

with all other relations being zero.

Before closing this section we discuss realization of fractional super algebras by "differential operators" in some linear spaces. Recall that for realization of super algebras one uses super derivatives which act on superspaces. Let $F(M)$ be an algebra of functions on a manifold M. For fixed grading n and the number N of "grassmannian" variables a fractional superspace is defined to be the direct product algebra $F(M) \times \Lambda^N_n$. We define fractional derivatives D_{θ_α} by the formulas

$$D_{\theta_\alpha} \theta_\beta = \delta_{\alpha\beta}, \qquad D_{\theta_\alpha}(ab) = D_{\theta_\alpha}(a) b + k(a) D_{\theta_\alpha}(b), \tag{98}$$

where $a,\ b \in \Lambda^N_n$ and

$$k(\theta_\alpha) = q\theta_\alpha, \qquad k(ab) = k(a)k(b). \tag{99}$$

Note that $D_{\theta_\alpha}(f) = 0$ and $k(f) = f$ if $f \in F(M)$. One can verify that these derivatives satisfy the relations

$$\sum_{\alpha_1 \ldots \alpha_n \in S_n} D_{\theta_{\alpha_1}} \cdots D_{\theta_{\alpha_n}} = 0 \tag{100}$$

By means of fractional derivatives and superspaces defined above one can construct a realization of a fractional superalgebras. For example the formulas

$$X_1 = -z^2 \frac{d}{dz} - zL, \quad X_2 = \frac{d}{dz}, \quad X_3 = 2z\frac{d}{dz} + L \tag{101}$$

$$Q_1 = D_\theta, \quad Q_2 = -zD_\theta, \quad Q_3 = \frac{q}{2}\theta^2 \frac{d}{dz}, \quad K = q^L, \tag{102}$$

where $q = e^{i\frac{2\pi}{3}}$ and

$$L = -q(2\theta^2 D_\theta^2 + D_\theta \theta^2 D_\theta) \tag{103}$$

define representation of the fractional algebra (iii) in the linear space $A(C) \times \Lambda_3^1$, where $A(C)$ is the algebra of polynomials of the complex variable z. Indeed using

$$L\theta^k = k\theta^k \tag{104}$$

and

$$\theta^2 D_\theta^2 + D_\theta^2 \theta^2 + D_\theta \theta^2 D_\theta = -q^2 \tag{105}$$

one can easily verify the relations (67), (81) and (97).

References

[1] M. Raush deTraunbenberg and M.J. Slupinski, *J. Math. Phys.*, **41**, 4556 (2000).

[2] M. Raush de Traunbenberg, Fractional sypersymmetry and Lie algebras, arXiv:hep-th/0007150.

[3] B. De Witt, "Supermanifols", Cambridge Univ. Press, Cambridge (1984).

[4] B. Kostant, "Graded manifolds, graded Lie theory and prequantization," in Lecture Notes in Mathematics, Spriger Verlag, New York, vol. 570, 177 (1977).

[5] M. Batchelor, *Tran. Am. Math. Soc.*, **258**, 257 (1980).

[6] N. Ya. Vilenkin and A. U. Klimyk, Representations of Lie Groups and Special Functions, vol 3, Kluwer Academic Press, The Netherland (1991).

[7] C. Ahn, D. Bernard and A. Leclair, *Nucl. Phys.* , **B346**, 409 (1990).

[8] M. Raush de Traunbenberg and M.J. Slupinski, *Mod. Phys. Lett. A*, **39**, 3051 (1997).

[9] R. Kerner, *J. Math. Phys.*, **33**, 403 (1992).

[10] V. Abramov, R. Kerner and B. Le Roy *J. Math. Phys.*, **38**, 1650 (1997).

[11] J.A. de Azcarraga and M.J. Macfarlane, *J. Math. Phys.*, **37**, 1115 (1996).

[12] S. Durand, *Mod. Phys. Lett.*, **A7**, 2905 (1992).

[13] H. Ahmedov and Ö.F. Dayi, *J. Phys. A*, **32**, 1895 (1999); *J. Phys. A*, **32**, 6247 (1999); *Mod. Phys. Lett. A*, **15**, no. 29, 1801 (2000); H. Ahmedov, *Turkish J. Phys.*, **24**, no. 3, 175 (2000)

[14] N. Ya. Vilenkin and A. U. Klimyk, Representations of Lie Groups and Special Functions, vol 3, Kluwer Academic Press, The Netherland (1991).